UNIVERSITY OF PENNSYLVANIA

CONGRUENCES OF CIRCLES STUDIED WITH REFERENCE TO THE SURFACE OF CENTERS

JOSEPH MILLER THOMAS

A THESIS

IN MATHEMATICS

PRESENTED TO THE FACULTY OF THE GRADUATE SCHOOL IN
PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY

PHILADELPHIA
1923

CONGRUENCES OF CIRCLES STUDIED WITH REFERENCE TO THE SURFACE OF CENTERS

A THESIS

IN MATHEMATICS

PRESENTED TO THE FACULTY OF THE GRADUATE SCHOOL OF THE UNIVERSITY OF PENNSYLVANIA IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

JOSEPH MILLER THOMAS

Reprint from
Annals of Mathematics, Vol. XXV, No. 3

PHILADELPHIA
1923

R. UNIV.
BIBLIOTHEEK
LEIDEN

CONGRUENCES OF CIRCLES STUDIED WITH REFERENCE TO THE SURFACE OF CENTERS.

By J. M. Thomas.

1. **Introduction.** Two parameter systems of circles which possess a single infinity of orthogonal surfaces have been extensively studied, especially by Ribaucour,* who named them cyclic systems. In particular, Ribaucour proved the following theorem: if in the tangent planes to a pseudospherical surface of curvature $-1/a^2$ circles of radius a be described with centers at the points of contact, these circles are the orthogonal trajectories of an infinity of surfaces of curvature $-1/a^2$. The process by which one of the orthogonal surfaces is obtained from the original pseudospherical surface has been called the transformation of Bianchi. This transformation has been extended by Bäcklund† to the case where the circles are cut at any constant angle by their trajectories. Beal‡ has further extended the transformation so that the angle is a function of the curvilinear coördinates on the original surface. The curvature is the same at corresponding points of the original and transformed surfaces. It satisfies a differential equation of the second order.

In all of the work mentioned above the framework of reference is the surface of centers only when the surface of centers coincides with the envelope of the planes of the circles. In the present paper are discussed certain properties possessed by a two parameter system of circles in connection with its surface of centers S, when the latter is distinct from the envelope of the planes. The lines of intersection of the planes of the circles with the tangent planes to S form a rectilinear congruence L. Since S is a focal surface of L, the edges of regression of one family of developable surfaces of L lie on S. It is therefore always possible to assemble the lines of L so that they are tangent to a single parameter family of curves Γ on S.

In § 2 the reference scheme to be used in this paper is explained. In § 3 certain equations of condition connected with the congruence of axes and the envelope of the planes are derived, and two theorems are proved

* "Mémoire sur la théorie générale des surfaces courbes", Journ. de Math., Ser. 4, vol. VII (1891), Chap. XI, XIII.

† Cf. Eisenhart, Differential Geometry, p. 284. Further reference to this text will be made in the form Eisenhart, p. 284.

‡ These Annals, 2d ser., vol. 17 (1916), p. 183 et seq.

to show interdependence among the congruence of axes, the curves Γ and the angle ψ between the plane of the circle and the normal to S.

The congruence of circles is next restricted so that each point P of a circle is on a surface S_1 which is one of a single parameter family of surfaces of the following nature. The tangent plane to S_1 passes through the center of the circle and makes an angle $90° - \Phi$ with the plane of the circle, Φ being a function of u and v, the curvilinear coördinates on S. Beal* has pointed out the existence of such congruences for a given surface of centers, and has studied in detail the case where the circles are in the tangent planes to S, that is, $\psi = 90°$. We call congruences of the above type for which ψ is not a right angle congruences C. In § 4 the fundamental equations for the existence of such congruences are derived and discussed.

In § 5 is proved the existence of congruences C in which all the circles have the same radius and the angle Φ is constant. Geometric interpretations of the three fundamental conditions are given for this case. It is found, for instance, that the circles must lie in the osculating planes of the curves Γ.

The cases where S degenerates into a curve or a point are discussed in § 6 and § 7 respectively.

The writer is greatly indebted to Professor F. W. Beal for assistance in the preparation of this paper.

2. With S we associate a set of moving rectangular axes, the x and y axes lying in the tangent plane and the z axis being the normal. The total displacements of a point (x, y, z) with respect to the moving axes are†

$$(1) \qquad \begin{cases} \delta x = dx + \xi\, du + \xi_1\, dv + (q\, du + q_1\, dv)\, z - (r\, du + r_1\, dv)\, y, \\ \delta y = dy + \eta\, du + \eta_1\, dv + (r\, du + r_1\, dv)\, x - (p\, du + p_1\, dv)\, z, \\ \delta z = dz \qquad\qquad + (p\, du + p_1\, dv)\, y - (q\, du + q_1\, dv)\, x. \end{cases}$$

In these formulas the translations ξ, ξ_1, η, η_1, and the rotations p, q, r, p_1, q_1, r_1 must satisfy the following conditions‡ which are equivalent to the Gauss and Codazzi equations for S:

$$(2) \qquad \begin{cases} p_v - p_{1u} = q r_1 - q_1 r, & \xi_v - \xi_{1u} = \eta r_1 - \eta_1 r, \\ q_v - q_{1u} = r p_1 - r_1 p, & \eta_v - \eta_{1u} = \xi_1 r - \xi r_1, \\ r_v - r_{1u} = p q_1 - p_1 q, & \xi q_1 - \xi_1 q = \eta p_1 - \eta_1 p. \end{cases}$$

---

* L. c., p. 182.

† Eisenhart, p. 170.

‡ Eisenhart, pp. 168, 170. Here, as throughout this paper, the subscripts u, v denote partial derivatives. Thus $p_v = \partial p / \partial v$.

If the x axis is chosen tangent to the curve $v = \text{const.}$ and an orthogonal system is parametric on S, then $\xi_1 = \eta = 0$, and the rotations have the following values*

$$(3)\quad \begin{cases} p = \bar{\omega}_u - \xi/\tau, & q = -\xi\cos\bar{\omega}/\varrho, & r = \xi\sin\bar{\omega}/\varrho, \\ p_1 = \eta_1\cos\bar{\omega}_1/\varrho_1, & q_1 = \bar{\omega}_{1v} - \eta_1/\tau_1, & r_1 = \eta_1\sin\bar{\omega}_1/\varrho_1, \end{cases}$$

where $1/\varrho$ is the curvature, $1/\tau$ the torsion and $\bar{\omega}$ the angle between the normal to S and the principal normal of $v = \text{const.}$, and the letters with the subscript 1 denote corresponding quantities for $u = \text{const.}$

2. **Congruences in general.** Taking $\xi_1 = \eta = 0$ and the curves Γ as $v = \text{const.}$, we next deduce certain properties of congruences in general. The axes of the circles form a rectilinear congruence. The coördinates of a point Q on the axis of a circle at distance t from its center are $(0, -t\cos\psi,\ t\sin\psi)$, and the direction cosines of the axis are proportional to these coördinates. To find the developables of the congruence of axes we express the condition that Q move tangent to the axis. It is

$$(4)\qquad \frac{\delta x}{0} = \frac{\delta y}{-\cos\psi} = \frac{\delta z}{\sin\psi}.$$

Making use of (1) we can replace (4) by the equivalent conditions

$$(5)\quad \begin{cases} \xi\,du + [(q\,du + q_1\,dv)\sin\psi + (r\,du + r_1\,dv)\cos\psi]\,t = 0, \\ \eta_1\sin\psi\,dv + (d\psi - p\,du - p_1\,dv)\,t = 0. \end{cases}$$

Elimination of t gives as the differential equation of the developables,

$$(6)\quad \xi(\psi_u - p)\,du^2 + (\xi\psi_v - \xi p_1 - q\eta_1\sin^2\psi - r\eta_1\sin\psi\cos\psi)\,du\,dv - \eta_1\sin\psi(q_1\sin\psi + r_1\cos\psi)\,dv^2 = 0.$$

The condition† that these correspond to a conjugate system on S can be reduced to

$$(7)\qquad p_1\psi_u - p\psi_v + (qr_1 - q_1 r)\sin\psi\cos\psi = 0.$$

Making use of the first of (2) and dividing by $\sin^2\psi$ we are led to

$$(8)\qquad \frac{\partial}{\partial v}(p\cot\psi) = \frac{\partial}{\partial u}(p_1\cot\psi).$$

* Eisenhart, p. 174.

† Eisenhart, pp. 130, 174.

The curves conjugate to $u = \text{const.}$ have for differential equation*

$$p\,du + p_1\,dv = 0.$$

Equation (8) expresses that $\cot\psi$ is an integrating factor of this last equation. Hence when (8) is true, the curves conjugate to the orthogonal trajectories of Γ can be found by a quadrature. Conversely, if the finite equation of these curves is known to be

$$f(u, v) = 0,$$

then the angle ψ can be determined so that (8) is satisfied by taking

$$\cot\psi = \frac{1}{p}\frac{\partial f}{\partial u} = \frac{1}{p_1}\frac{\partial f}{\partial v}.$$

If the lines of curvature are parametric, $p = q_1 = 0$ and (8) can be written

$$p_1 \cot\psi = V, \tag{9}$$

where V is a function of v alone. If now $r_1 = 0$, it is seen from (3) that the curves $u = \text{const.}$ are geodesics. Being also lines of curvature, they are plane. From the first equation of (2) it follows that p_1 is a function of v alone and hence from (9) that ψ is likewise independent of u. Reasoning from the same equations in a similar fashion we find we can state the following

THEOREM 1. *If the curves Γ are lines of curvature and S is not developable, each of the following statements is a consequence of the remaining two:*

(i) *The developables of the congruence of axes correspond to a conjugate system on S.*

(ii) *The orthogonal trajectories of the curves Γ are plane geodesics.*

(iii) *The inclination of the plane of the circle to the normal of S is constant along each curve Γ.*

By eliminating $du : dv$ from equations (5) we obtain

$$\begin{aligned}\xi\eta_1 \sin\psi + (\xi\psi_v - \xi p_1 + q\eta_1 \sin^2\psi + r\eta_1 \sin\psi\cos\psi)\,t \\ + [(q\psi_v - q_1\psi_u + pq_1 - p_1 q)\sin\psi \\ + (r\psi_v - r_1\psi_u - rp_1 + r_1 p)\cos\psi]\,t^2 = 0.\end{aligned} \tag{10}$$

The roots t_1, t_2 of this equation are the distances to the focal points.

* Eisenhart, p. 173.

If on the other hand Q is to move normal to the axis, we must have

$$\delta y \cos\psi - \delta z \sin\psi = 0,$$

or

$$dt = \eta_1 \cos\psi \, dv.$$

The condition that the congruence of axes be normal is therefore

$$\eta_1 \cos\psi = V, \tag{11}$$

where V denotes a function of v alone. If $r_1 = 0$, it follows from (2) that η_1 is independent of u and from (11) that the same is true of ψ. We can in this manner prove

THEOREM 2. *Each of the following statements is a consequence of the other two*:

(i) *The congruence of axes of the circles is normal.*

(ii) *The orthogonal trajectories of the curves Γ are geodesics.*

(iii) *The inclination of the plane of the circle to the normal of S is constant along each curve Γ.*

We next find the coördinates (x_0, y_0, z_0) of the point of contact of the plane of the circle with its envelope. The condition on the coördinates is

$$\delta y_0 \cos\psi - \delta z_0 \sin\psi = 0.$$

Putting

$$y_0 = \lambda \sin\psi, \qquad z_0 = \lambda \cos\psi,$$

substituting the values of δy_0 and δz_0 from (1) and equating the coefficients of du and dv to zero we get

$$\begin{aligned} (q \sin\psi + r \cos\psi) x_0 + (\psi_u - p)\lambda &= 0, \\ (q_1 \sin\psi + r_1 \cos\psi) x_0 + (\psi_v - p_1)\lambda &= -\eta_1 \cos\psi. \end{aligned}$$

If we let $\mu = 1/t_1 t_2$ and make use of (10) we can put the solution of these equations in the form

$$x_0 = (\psi_u - p) \cot\psi/\xi\mu, \qquad \lambda = -(q \sin\psi + r \cos\psi) \cot\psi/\xi\mu. \tag{12}$$

3. **Congruences** C. If R denotes the radius of a circle and $\varkappa$ the angle made by the corresponding line of L with the x axis of the trihedral, the coördinates of a point P on the circle are

$$(13)\quad R(\cos\theta\cos\varkappa-\sin\theta\sin\psi\sin\varkappa),\qquad R(\cos\theta\sin\varkappa+\sin\theta\sin\psi\cos\varkappa),$$
$$R\sin\theta\cos\psi,$$

where θ denotes the angle made by the radius to P with the line L. The condition that P generate a surface S_1, as defined in the introduction, is

$$(14)\qquad l\delta x+m\delta y+n\delta z=0,$$

where l, m, n are the direction cosines of the normal to S_1 and have the values

$$(15)\quad\begin{cases} l &= \sin\theta\cos\Phi\cos\varkappa+(\sin\Phi\cos\psi+\cos\theta\cos\Phi\sin\psi)\sin\varkappa,\\ m &= \sin\theta\cos\Phi\sin\varkappa-(\sin\Phi\cos\psi+\cos\theta\cos\Phi\sin\psi)\cos\varkappa,\\ n &= \sin\Phi\sin\psi-\cos\theta\cos\Phi\cos\psi.\end{cases}$$

When substitutions are made from (1), (13) and (15), equation (14) becomes

$$(16)\qquad A\,d\theta+B\,du+C\,dv=0,$$

where

$$\begin{aligned} A &= -R\cos\Phi,\\ B &= (\xi\cos\varkappa+\eta\sin\varkappa)\sin\theta\cos\Phi\\ &\quad+(\xi\sin\varkappa-\eta\cos\varkappa)(\cos\theta\cos\Phi\sin\psi+\sin\Phi\cos\psi)\\ &\quad+R(p\cos\varkappa+q\sin\varkappa-\psi_u)\sin\theta\sin\Phi\\ &\quad+R(p\sin\varkappa-q\cos\varkappa)(\cos\theta\sin\Phi\sin\psi-\cos\Phi\cos\psi)\\ &\quad+R(r+\varkappa_u)(\cos\theta\sin\Phi\cos\psi+\cos\Phi\sin\psi),\end{aligned}$$

and C can be obtained from B by adding the subscript 1 to the translations and rotations and changing the subscript u into v. Equation (16) can be made to assume the Riccati form by the substitution $\theta=2\tan^{-1}t$.

The necessary and sufficient condition for a congruence C is that equation (16) possess a solution $\theta(u,v)$ involving an arbitrary parameter, or that the condition of integrability of (16), namely,

$$A\left(\frac{\partial C}{\partial u}-\frac{\partial B}{\partial v}\right)+B\left(\frac{\partial A}{\partial v}-\frac{\partial C}{\partial\theta}\right)+C\left(\frac{\partial B}{\partial\theta}-\frac{\partial A}{\partial u}\right)=0,$$

be identically satisfied. This condition assumes the form

$$L\sin\theta+M\cos\theta+N=0,$$

where L, M, and N are independent of θ. The equation in θ is satisfied by not more than two values unless $L = M = N = 0$, when it is identically satisfied. The fundamental equations for the existence of a congruence C are accordingly

$$
\begin{aligned}
(17)\quad & R(\xi p_1 - \xi_1 p)\sin\psi\cos\psi\sin^2\varkappa - R(\xi q_1 - \xi_1 q)\sin 2\psi\sin\varkappa\cos\varkappa \\
& - R(\xi r_1 - \xi_1 r)\cos^2\psi\sin\varkappa + R(\eta q_1 - \eta_1 q)\sin\psi\cos\psi\cos^2\varkappa \\
& + R(\eta r_1 - \eta_1 r)\cos^2\psi\cos\varkappa - (\xi R_v - \xi_1 R_u)\cos^2\Phi\cos\varkappa \\
& - (\eta R_v - \eta_1 R_u)\cos^2\Phi\sin\varkappa + R^2(\psi_v\Phi_u - \psi_u\Phi_v) \\
& + R^2(p\Phi_v - p_1\Phi_u)\cos\varkappa + R^2(q\Phi_v - q_1\Phi_u)\sin\varkappa \\
& - R(\xi\varkappa_v - \xi_1\varkappa_u)\cos^2\psi\sin\varkappa + R(\eta\varkappa_v - \eta_1\varkappa_u)\cos^2\psi\cos\varkappa = 0,
\end{aligned}
$$

$$
\begin{aligned}
(18)\quad & (\xi\eta_1 - \xi_1\eta)\sin\Phi\cos\Phi\cos\psi + R(\xi p_1 - \xi_1 p)\cos\psi\sin\varkappa\cos\varkappa \\
& - R(\xi q_1 - \xi_1 q)\cos\psi\cos 2\varkappa - R(\eta q_1 - \eta_1 q)\cos\psi\sin\varkappa\cos\varkappa \\
& + (\xi R_v - \xi_1 R_u)\cos^2\Phi\sin\psi\sin\varkappa - (\eta R_v - \eta_1 R_u)\cos^2\Phi\sin\psi\cos\varkappa \\
& - R(\xi\psi_v - \xi_1\psi_u)\cos\psi\sin\varkappa + R(\eta\psi_v - \eta_1\psi_u)\cos\psi\cos\varkappa \\
& - R^2(p\Phi_v - p_1\Phi_u)\sin\psi\sin\varkappa + R^2(q\Phi_v - q_1\Phi_u)\sin\psi\cos\varkappa \\
& + R^2(r\Phi_v - r_1\Phi_u)\cos\psi + R^2(\Phi_v\varkappa_u - \Phi_u\varkappa_v)\cos\psi = 0,
\end{aligned}
$$

$$
\begin{aligned}
(19)\quad & (\xi\eta_1 - \xi_1\eta)\cos^2\Phi\sin\psi + R^2(qr_1 - q_1 r)\cos\psi\sin\varkappa \\
& - R^2(rp_1 - r_1 p)\cos\psi\cos\varkappa + R^2(pq_1 - p_1 q)\sin\psi \\
& + (\xi R_v - \xi_1 R_u)\sin\Phi\cos\Phi\cos\psi\sin\varkappa \\
& - (\eta R_v - \eta_1 R_u)\sin\Phi\cos\Phi\cos\psi\cos\varkappa - R^2(p\psi_v - p_1\psi_u)\sin\psi\sin\varkappa \\
& + R^2(q\psi_v - q_1\psi_u)\sin\psi\cos\varkappa + R^2(r\psi_v - r_1\psi_u)\cos\psi \\
& + R^2(\psi_v\varkappa_u - \psi_u\varkappa_v)\cos\psi - R(\xi\Phi_v - \xi_1\Phi_u)\cos\psi\sin\varkappa \\
& + R(\eta\Phi_v - \eta_1\Phi_u)\cos\psi\cos\varkappa + R^2(p\varkappa_v - p_1\varkappa_u)\cos\psi\cos\varkappa \\
& + R^2(q\varkappa_v - q_1\varkappa_u)\cos\psi\sin\varkappa = 0.
\end{aligned}
$$

In discussing this system of equations we can without loss of generality put $\xi_1 = \eta = p = q_1 = 0$. This amounts to taking the lines of curvature parametric on S and the coördinate axes tangent to them. Since the equations can be solved for the u derivatives of three of the four quantities R, Φ, ψ and $\varkappa$ the following statements are true in general:

(i) Any one of the functions R, Φ, ψ and $\varkappa$, or one relation among them, can be chosen arbitrarily. Thus, choosing $\Phi = 0$, we see that any sur-

face can be the surface of centers of a cyclic system. Similarly the geometric property given by any one of equations (8), (9), (11), (27), etc., can be imposed without restricting S.

(ii) The function $\varkappa$ determines the family of curves Γ when the moving trihedral has been chosen. For any family of curves Γ on any surface S there exist functions R, Φ and ψ which associate with S a congruence C.

Equations (17), (18) and (19) become for $\varkappa = 0$

$$(20)\quad \begin{aligned} &R^2(\psi_u \Phi_v - \psi_v \Phi_u) - R^2(p\,\Phi_v - p_1\,\Phi_u) - R\cos^2\psi(\eta\, r_1 - \eta_1 r) \\ &\quad - R\sin\psi\cos\psi(\eta\, q_1 - \eta_1 q) + \cos^2\Phi(\xi\, R_v - \xi_1\, R_u) = 0, \end{aligned}$$

$$(21)\quad \begin{aligned} &R^2\sin\psi(q\,\Phi_v - q_1\,\Phi_u) + R^2\cos\psi(r\,\Phi_v - r_1\,\Phi_u) \\ &\quad + R\cos\psi(\eta\,\psi_v - \eta_1\,\psi_u) - R\cos\psi(\xi\, q_1 - \xi_1 q) \\ &\quad + \sin\Phi\cos\Phi\cos\psi(\xi\,\eta_1 - \xi_1\,\eta) - \cos^2\Phi\sin\psi(\eta\, R_v - \eta_1\, R_u) = 0, \end{aligned}$$

$$(22)\quad \begin{aligned} &R^2\sin\psi(q\,\psi_v - q_1\,\psi_u) + R^2\cos\psi(r\,\psi_v - r_1\,\psi_u) + R^2\sin\psi(p\,q_1 - p_1 q) \\ &\quad - R^2\cos\psi(r p_1 - r_1 p) + R\cos\psi(\eta\,\Phi_v - \eta_1\,\Phi_u) \\ &\quad + \cos^2\Phi\sin\psi(\xi\,\eta_1 - \xi_1\,\eta) - \sin\Phi\cos\Phi\cos\psi(\eta\, R_v - \eta_1\, R_u) = 0. \end{aligned}$$

If R, Φ and ψ are given functions, then functions ξ, η, ξ_1, η_1, p, q, r, p_1, q_1, r_1 satisfying (2), (20), (21), and (22) can be shown to exist in general. For equations (20), (21), (22) and the last of (2) can be solved for ξ, η, p and q. If the results are substituted in equations (2), the latter can be solved for the u derivatives of p_1, q_1, r_1, ξ_1 and η_1 leaving r arbitrary, and the results are in a form to which Cauchy's theorem* can be applied to show the existence of the functions involved. Hence solutions, and consequently surfaces S, exist in general, but in particular cases a determinant involved in the solution may vanish and cause the method to fail. Moreover, even when the functions exist the corresponding surfaces S may be degenerate as is seen below in § 5 when R, Φ and ψ are all taken constant.

With $\varkappa$ still equal to zero, we next find the curvature of the surfaces S_1. If we take a point on the normal to S_1 at distance w from P, and express the condition that it move in the direction given by equations (15), we get two equations in $du : dv$. Elimination of this derivative gives the following equation to determine w:

$$(23)\qquad Pw^2 + Qw + T = 0,$$

* Jordan, Cours d'Analyse, (1896), t. III, p. 305.

where

$$\begin{aligned}P = \sin\theta\cos^2\Phi\,[&\Phi_v\xi - \Phi_u\xi_1 + (\psi_v\xi - \psi_u\xi_1)\cos\theta\\ &-(\xi q_1 - \xi_1 q)\sin\theta\sin\psi - (\xi p_1 - \xi_1 p)\cos\theta - (\xi r_1 - \xi_1 r)\sin\theta\cos\psi]\\ &-\cos\Phi(\sin\Phi\cos\psi + \cos\theta\cos\Phi\sin\psi)\,[\Phi_v\eta - \Phi_u\eta_1\\ &+(\psi_v\eta - \psi_u\eta_1)\cos\theta - (\eta p_1 - \eta_1 p)\cos\theta\\ &-(\eta q_1 - \eta_1 q)\sin\theta\sin\psi - (\eta r_1 - \eta_1 r)\sin\theta\cos\psi],\end{aligned}$$

$$\begin{aligned}T = -R^2\sin\theta\sin\psi\,&[(\eta\psi_v - \eta_1\psi_u)\sin\theta - (\eta p_1 - \eta_1 p)\sin\theta\\ &+(\eta q_1 - \eta_1 q)\cos\theta\sin\psi + (\eta r_1 - \eta_1 r)\cos\theta\cos\psi]\\ &-R^2\cos\theta\,[(\xi\psi_v - \xi_1\psi_u)\sin\theta - (\xi p_1 - \xi_1 p)\sin\theta\\ &+(\xi q_1 - \xi_1 q)\cos\theta\sin\psi + (\xi r_1 - \xi_1 r)\cos\theta\cos\psi]\\ &-R(\xi\eta_1 - \xi_1\eta)\cos\theta\cos\psi - R^2\sin\theta\,(\psi_v R_u - \psi_u R_v)\\ &+R\cos\psi\,(\eta R_v - \eta_1 R_u) - R^2\sin\theta\,(p R_v - p_1 R_u)\\ &+R^2\cos\theta\sin\psi\,(q R_v - q_1 R_u) + R^2\cos\theta\cos\psi\,(r R_v - r_1 R_u),\end{aligned}$$

and Q does not contain w. The curvature K_1 of the surfaces S_1 is the reciprocal of the product of the roots of (23), that is

$$K_1 = P/T. \tag{24}$$

This expression is not in general independent of θ, but when $\psi = 90^0$, it reduces to

$$K_1 = -\cos^2\Phi/R^2,$$

and all of the transformed surfaces S_1 have the same total curvature as the original surface S.* The derivative of K_1 with respect to θ equated to zero yields an equation of the fourth degree in $\sin\theta$ and of the first in $\cos\theta$. This equation gives the necessary condition for transformed surfaces of maximum or minimum curvature at a point. It is not satisfied by more than eight independent values of θ unless it is identically satisfied, in which case the curvature is independent of θ.

THEOREM 3. *The curvature of the surfaces S_1 cannot have more than eight extreme values at a point of general position on S.*

The direction cosines of the planes normal to S_1 and passing through one of the corresponding focal points of the congruence of axes are readily found to be proportional to

* This result is given by Beal, loc., cit., p. 184.

$$t_i \cos\theta \cos\Phi - R \sin\theta \sin\Phi,$$

$$t_i \sin\theta \cos\Phi \sin\psi - R \cos\Phi \cos\psi + R \cos\theta \sin\Phi \sin\psi,$$

$$t_i \sin\theta \cos\Phi \cos\psi + R \cos\Phi \sin\psi + R \cos\theta \sin\Phi \cos\psi, \quad (i = 1, 2),$$

where the t's satisfy (10). The condition that the two planes be perpendicular can be reduced to

$$1/t_1 t_2 + \cos^2\Phi/R^2 = 0. \tag{25}$$

This condition remains the same whatever the value of $\varkappa$ or ψ. By the use of (10), equation (22) can be written in the form ($\xi_1 = \eta = 0$)

$$1/t_1 t_2 + \cos^2\Phi/R^2 = \cot\psi(R\Phi_u - R_u \sin\Phi \cos\Phi)/\xi R^2.$$

The necessary and sufficient condition that condition (25) be satisfied by a congruence C is therefore

$$\cot\psi(R\Phi_u - R_u \sin\Phi \cos\Phi) = 0. \tag{26}$$

When $\cot\psi$ is divided out* and the equation is integrated, (26) becomes

$$R \cot\Phi = V, \tag{27}$$

where V is a function of v alone. Conversely, if (27) is satisfied, (25) is also, unless $\psi = 0$. In the exceptional case one focal point is at the origin and the other is the point at infinity on the y axis. It is readily seen geometrically that the two planes in question are still perpendicular. The solution $\Phi = 0$ of (26) is not given by (27). It corresponds to a cyclic system, for which it is well known† that the joins of a point on the circle to the corresponding focal points, and consequently the above normal planes to S_1, are perpendicular. We have therefore

THEOREM 4. *For a congruence C which is not a cyclic system the necessary and sufficient condition for the perpendicularity of the two planes which are normal to S_1 and pass through the corresponding focal points of the congruence of axes is that $R \cot\Phi$ be constant along each curve Γ.*

* $\cot\psi \neq 0$ for a congruence C. If $\psi = 90^0$, equation (25) is satisfied. Beal, loc. cit. p. 183, has remarked that an equivalent geometric property is possessed by his congruences C_0.

† Eisenhart, p. 437.

In the case of cyclic systems, the formulas (12) can be written in a compact form. Equations (20) and (21) become for $\Phi = 0$, $\xi_1 = \eta = 0$

$$R_u/R = (\psi_u - p)\cot\psi, \qquad R_v/R = -\eta_1 \cos\psi\,(q\sin\psi + r\cos\psi)/\xi.$$

From (25) we have

$$\mu = -1/R^2.$$

Hence formulas (12) can be written

$$x_0 = -\frac{1}{2}\frac{dR^2}{ds}, \qquad y_0 = -\frac{1}{2}\frac{dR^2}{ds_1}, \qquad z_0 = -\frac{1}{2}\frac{dR^2}{ds_1}\cot\psi,$$

where s and s_1 denote the arc of the curves Γ and of their orthogonal trajectories respectively.

4. **Congruences $C_{R\Phi}$.** A congruence C for which R and Φ are constant* will be denoted by $C_{R\Phi}$, the subscripts indicating the quantities which are kept constant. The fundamental equations for such a congruence are ($\xi_1 = \eta = 0$)

(28) $$\tan\psi = -r/q,$$

(29) $$\psi_u = p + a\xi,$$

(30) $$b\eta_1 = q_1 + r_1\cot\psi,$$

where a and b are constants defined by

(31) $$a = \sin\Phi\cos\Phi/R,$$

(32) $$b = \cot\Phi/R.$$

From (3) and (28) it is seen that ψ can be taken equal to $\bar{\omega}$. Hence the circles lie in the osculating planes of the curves Γ. Equation (29) can be rewritten as

$$\bar{\omega}_u = p + a\xi.$$

* The case $\phi = 90°$ is, of course, excluded. If $\phi = 90°$ and R and Φ are constant, S is a pseudospherical surface, as is seen from (25). The surfaces S_1 are its Bäcklund transforms. Cf. Beal, loc. cit., p. 185. Also, Eisenhart, p. 284.

By use of the first of equations (3) we get $1/\tau = a$. The torsion of the curves Γ is therefore an absolute constant. The geometric interpretation of (30) is given by (25) or by theorem 4. We have therefore

THEOREM 5. *The circles of a congruence $C_{R\Phi}$ lie in the osculating planes of the curves Γ, all of which are of the same constant torsion and have the product of the distances to the focal points of their congruence of binormals constant.*

From (20) is readily proved

THEOREM 6. *The necessary and sufficient condition that the circles of a congruence C_Φ lie in the osculating planes of the curves Γ is that R be constant along each of their orthogonal trajectories.*

To show the existence of surfaces S with which are associated congruences $C_{R\Phi}$, it is necessary to establish the existence of functions satisfying equations (2), (28), (29) and (30). To this end eliminate p, r, q_1 and r_1 by means of (28), (29), (30) and the last of (2). The resulting equations can be combined so as to give an expression for q in which the only derivatives are ψ_u and ψ_v. When q is eliminated by this relation and the substitutions

$$u_1 = u + v, \qquad v_1 = u - v \tag{33}$$

are made, there results a system which can be solved for ξ_{u_1}, η_{1u_1}, p_{1u_1}, and $\psi_{u_1u_1}$. Moreover the expressions for these derivatives are of the form to which Cauchy's existence theorem* can be applied. Hence surfaces S exist. They do not degenerate into curves since the linear element is in the orthogonal form.

If we further impose the restriction that ψ be constant, introduce the new variables

$$\alpha = 1/\varrho, \qquad \beta = 1/\varrho_1, \qquad \gamma = 1/\tau_1,$$

substitute from (3) in (2), (28), (29) and (30), and combine the resulting equations, we find that the conditions for the existence of the corresponding congruence are

$$\xi_v = -\xi\eta_1\alpha\sin\psi, \tag{34}$$

$$\eta_{1u} = c\xi\eta_1, \tag{35}$$

$$\beta\sin\bar{\omega}_1 = c, \tag{36}$$

$$\bar{\omega}_{1v} = \eta_1(\gamma + a), \tag{37}$$

* Jordan, loc. cit.

$$\xi(a+b)\alpha\sin\psi - \xi c\beta\cos\bar{\omega}_1 - (\beta\cos\bar{\omega}_1)_u = 0, \tag{38}$$

$$\alpha_v\cos\psi = \eta_1(\alpha^2\sin\psi\cos\psi - 2ac - \alpha\beta\sin\psi\cos\bar{\omega}_1), \tag{39}$$

$$\alpha_v\sin\psi = \eta_1(\alpha^2\sin^2\psi + c^2 - a^2 + \alpha\beta\cos\psi\cos\bar{\omega}_1), \tag{40}$$

c being a constant defined by the equation

$$c = (b-a)\tan\psi. \tag{41}$$

From (36) follows*

$$(\beta\cos\bar{\omega}_1)_u = -c\csc^2\bar{\omega}_1\,\bar{\omega}_{1u};$$

and from (38)

$$\bar{\omega}_{1u} = \xi c\sin\bar{\omega}_1\cos\bar{\omega}_1 - (a+b)\xi\alpha\sin\psi\sin^2\bar{\omega}_1/c. \tag{42}$$

Multiplying (39) by $\sin\psi$, (40) by $\cos\psi$ and subtracting, we get

$$\alpha\beta\cos\bar{\omega}_1 = -2ac\sin\psi + (a^2-c^2)\cos\psi = k, \tag{43}$$

where k is constant. Thus we have

$$\alpha = k\tan\bar{\omega}_1/c, \tag{44}$$

by the use of which equation (42) becomes

$$\bar{\omega}_{1u} = \xi c\sin\bar{\omega}_1\cos\bar{\omega}_1 - k(a+b)\xi\sin\psi\sin^2\bar{\omega}_1\tan\bar{\omega}_1/c^2. \tag{45}$$

If we differentiate (44) and use (39) or (40), we get

$$\bar{\omega}_{1v} = \eta_1(k\sin^2\bar{\omega}_1\sin\psi/c - 2ac^2\cos^2\bar{\omega}_1/k\cos\psi - c\tan\psi\cos\bar{\omega}_1). \tag{46}$$

Expressing the condition

$$\bar{\omega}_{1uv} = \bar{\omega}_{1vu},$$

and using (34), (35), (44), (45) and (46), we find

$$\xi\eta_1\cos^3\Phi\sin^2\psi\sin^2\bar{\omega}_1 = 0. \tag{47}$$

If we have $\xi\eta_1 = 0$, it is seen from (34) and (35) that the linear element of S assumes one of the forms

$$ds^2 = Udu^2 \qquad \text{or} \qquad ds^2 = Vdv^2,$$

---

*If $\sin\bar{\omega}_1 = 0$, then from (36) $c = 0$ and either $a = b$, in which case $\Phi = 90°$ and S and its transforms become a plane; or $\psi = 0$. We exclude $c = 0$ for the present.

where U and V are functions of u and v respectively, and that S is therefore a curve. The congruence C degenerates into a single parameter family of circles.

From (47) it is therefore seen that the only possible value for ψ is zero.* We have therefore proved

THEOREM 7. *In a congruence $C_{R\Phi}$ the only possible constant value for ψ is zero.*

Since (36) states that the curves $u = \text{const.}$ all have the same constant geodesic curvature, we can express the above result as follows: there exists no curved surface which is the locus of a single parameter family of curves all of which have the same constant torsion, all of whose principal normals make the same constant oblique angle with the corresponding normals to the surface, and whose orthogonal trajectories all have the same constant geodesic curvature.

With $\psi = 0$ and R and Φ constant the fundamental equations become

$$r = 0, \tag{48}$$

$$Rq_1 - \eta_1 \sin\Phi \cos\Phi = 0, \tag{49}$$

$$r_1 p = 0. \tag{50}$$

From (48) the curves Γ are geodesics. If from (50) we choose $r_1 = 0$, S has an orthogonal system of geodesics and is therefore developable. The fundamental quantities for S are then

$$\xi = U, \quad \eta_1 = V, \quad \xi_1 = \eta = r = r_1 = 0,$$
$$p = -aU, \quad p_1 = akV, \quad q = -aU/k, \quad q_1 = aV,$$

where U and V are arbitrary functions of u and v respectively, k is an arbitrary constant, and a is defined by (31).

Choosing $p = 0$, we get from the last equation in (2) $q_1 = 0$. The curves Γ are therefore lines of curvature and, being also geodesics, are plane. S is a surface of Monge.† Finally (49) forces us to choose $\Phi = 0$. The system is therefore cyclic.

THEOREM 8. *If in a congruence $C_{R\Phi}$ whose surface of centers S is not developable the plane of each circle passes through the corresponding normal to S, the circles form a cyclic system and S is a surface of Monge on which the curves Γ are plane geodesics.*

* See preceding foot-note.

† Eisenhart, p. 306.

A particular example of this last type of congruence is furnished by the pseudospherical surface of Monge for which the fundamental quantities are

$$\xi_1 = \eta = p = q_1 = 0, \qquad \xi = a\cos\omega, \qquad \eta_1 = a\sin\omega,$$
$$p_1 = \cos\omega, \qquad q = \sin\omega, \qquad r = \omega_v = 0, \qquad r_1 = \omega_u = \sin\omega.$$

The normal to S_1 is the tangent to the circle at P. Analysis similar to that employed above in obtaining the total curvature of the surfaces S_1 leads to the following values for w_1 and w_2, the principal radii of curvature of S_1,

$$w_1 = -R\cot\theta, \qquad w_2 = -R\tan(\omega-\theta) - a\sin\omega\sec(\omega-\theta).$$

It is easy to give geometric constructions for the principal centers of curvature: the normal to S_1 meets the z axis in the first and the line

$$y = 0, \qquad z = (x+a)\tan\omega \tag{51}$$

in the second. Hence for a given point the principal centers of curvature for all the transformed surfaces S_1 lie on a pair of lines making an angle $90^\circ - \omega$ with each other. This line (51) is the characteristic of the plane of the circle, and the point of contact of this plane with its envelope is the intersection of (51) with the x axis. If from the intersection of (51) with the z axis, we draw tangents to the circle, we get the normals to two transformed surfaces which have the points of contact for umbilical points.

5. **S is a curve.** If the surface of centers reduces to a curve, we can take for the moving triedral the tangent, principal normal and binormal of the curve and for parameter u its arc. With this choice, the translations and rotations have the following values:

$$\xi = 1, \quad \xi_1 = \eta = \eta_1 = q = p_1 = q_1 = r_1 = 0, \quad p = -1/\tau, \quad r = 1/\rho. \tag{52}$$

From the fundamental equations for a congruence C we see as before that in general we can choose a relation between R, Φ, ψ and $\varkappa$, or any one of them, without restricting the curve of centers. With $\Phi = 0$, however, the equations become

$$R_v\cos\varkappa + R\varkappa_v\cos^2\psi\sin\varkappa = 0, \tag{53}$$

$$\sin\varkappa(R_v\sin\psi - R\psi_v\cos\psi) = 0, \tag{54}$$

$$r\psi_v + \psi_v\varkappa_u - \psi_u\varkappa_v + p\varkappa_v\cos\varkappa - p\psi_v\tan\psi\sin\varkappa = 0. \tag{55}$$

One set of solutions of these equations is $\sin\psi = \cos\varkappa = 0$. The circles are then in the normal planes to the curve of centers. R is unrestricted.

If $\sin\varkappa$ equals zero, it follows from (53) and (55) either that we have $R_v = \psi_v = 0$, in which case there is only a single infinity of circles; or that r is zero, in which case the curve of centers is a straight line. If we have $\varkappa_v = 0$, $\sin\varkappa \neq 0$ and $\cos\varkappa \neq 0$, (53) and (54) show that there is only a single infinity of circles.

If $\varkappa_v$ and r do not vanish, then ψ is zero. For assuming $\psi \neq 0$, we get from (53)

$$\tan\psi = U\cos\varkappa. \tag{56}$$

Substitution in (55) gives

$$\varkappa_v[(p + pU^2 - U')\cos\varkappa - rU\sin\varkappa] = 0.$$

This gives under the assumptions $U = 0$ and hence from (56), $\psi = 0$. Equations (53), (54) and (55) can be replaced for $\psi = 0$ by

$$R = U\cos\varkappa, \tag{57}$$

$$p = 0. \tag{58}$$

From (58) and (52), it follows that the curve of centers is plane.

We can state therefore

THEOREM 9. *If the centers of the circles of a cyclic system lie on a curve, then*

(i) *The circles lie in the normal planes to the curve of centers and their radii are given by an arbitrary function*; *or*

(ii) *The center locus is a straight line*; *or*

(iii) *The curve of centers is plane and the circles lie in planes through the binormals.*

A cyclic system with a given plane curve for locus of centers can be constructed in the following way. Choose an arbitrary function U of the arc. At each point of the curve construct the circles

$$x^2 + y^2 = \pm Ux,\ z = 0. \tag{59}$$

Any circle which lies in a plane perpendicular to the plane of the given curve and which has for diameter a line segment passing through the origin and terminating in the two circles (59) is a circle of the congruence.

When combined with (52), the fundamental equations (20), (21) and (22) become, $\varkappa$ being zero,

$$R^2(\psi_v \Phi_u - \psi_u \Phi_v) - R_v \cos^2 \Phi + R^2 p \Phi_v = 0, \tag{60}$$

$$r \Phi_v \cos \psi = 0, \tag{61}$$

$$r \psi_v \cos \psi = 0. \tag{62}$$

The assumption $r = 0$ leads to a straight line. The choice $\Phi_v = \psi_v = 0$ gives $R_v = 0$ from (60). Hence if there is a double infinity of circles and the center locus is not a straight line, ψ is 90° and the circles lie in the osculating planes. Equation (60) when integrated for this value of ψ becomes

$$1/R + p \tan \Phi = U, \tag{63}$$

where U is an arbitrary function of u.

6. S **is a point.** The triehedral for the surface is indeterminate, so ψ can be taken equal to 90°. Moreover, the translations all vanish. Hence the fundamental equations are

$$p \Phi_v - p_1 \Phi_u = 0, \qquad q \Phi_v - q_1 \Phi_u = 0, \qquad p q_1 - p_1 q = 0.$$

The last equation states that the circles lie in a single infinity of planes. The above system can be replaced by

$$\Phi_u = e^f p, \tag{64}$$

$$\Phi_v = e^f p_1, \tag{65}$$

$$p q_1 - p_1 q = 0. \tag{66}$$

The condition $\Phi_{uv} = \Phi_{vu}$ gives

$$f_u = (q r_1 - q_1 r + p f_v)/p_1. \tag{67}$$

Equation (66) is satisfied by the rotations for any developable surface. With such a choice of p, p_1, q and q_1 there exists a solution of (67) which when substituted in (64) and (65) gives Φ. There is no restriction on the radius. If Φ is constant, the only condition is (66).

UNIVERSITY OF PENNSYLVANIA,
PHILADELPHIA, PENNSYLVANIA.

Printed by Lütcke & Wulff, Hamburg, Germany